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The Liouville-type theorem for integrable Hamiltonian systems with incomplete flows

For integrable Hamiltonian systems with two degrees of freedom whose Hamiltonian vector fields have incomplete flows, an analogue of the Liouville theorem is established. A canonical Liouville fibration is defined by means of an "exact" 2-parameter family of flat polygons equipped with certain pairing of sides. For the integrable Hamiltonian systems given by the vector field $v = (-\partial f/\partial w, \partial f/\partial z)$ on $\mathbb{C}^2$ where $f = f(z, w)$ is a complex polynomial in 2 variables, geometric properties of Liouville fibrations are described.

Key words: integrable Hamiltonian system, Liouville theorem, incomplete flows, Newton polygon.

MSC: 37J05, 37J35.

§ 1. Introduction

Suppose (M^{2k}, ω) is a symplectic manifold and $F = (f_1, \dots, f_k) : M^{2k} \to \mathbb{R}^k$ is a set of (functionally independent) smooth functions in involution. The triple (M^{2k}, ω, f_1) is called an *integrable Hamiltonian system*, and the triple $L = (M^{2k}, \omega, F)$ is called a *Liouville fibration*. Two Liouville fibrations L, L_1 are called *isomorphic* if $\omega = h^*\omega_1$ and $F = F_1 \circ h$, for some diffeomorphism $h : M^{2k} \to M_1^{2k}$.

The classical Liouville theorem describes a Liouville fibration in a small neighbourhood of any compact connected regular fibre $T_\xi = F^{-1}(\xi)$ up to isomorphism. The works [1, 2, 3] stated the problem of finding an analogue of the Liouville theorem for integrable Hamiltonian systems whose Hamiltonian vector fields have incomplete flows. We solve this problem for systems with $k = 2$ degrees of freedom (theorem 3). We define a "canonical" Liouville fibration $L_{W;U,d\mathbf{I},\mathbf{S};[\Theta]}$ by means of so-called *combinatorial-geometrical-topological data* (shortly *CGT-data*), denoted by $(W; U, d\mathbf{I}, \mathbf{S}; [\Theta])$, which are a 2-parameter family of flat polygons with certain pairing of sides. Dynamical properties of the corresponding Hamiltonian flow on an individual "generic" fibre T_ξ were studied by many authors (see [4, 5, 6] and references therein).

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§ 2. Combinatorial-geometrical-topological data

The CGT-data are defined as follows.

2.1. Combinatorial data. Suppose $n \in \mathbb{N}$, $\sigma \in \Sigma_{2n}$ is a permutation of the set $\{1,\dots,2n\}$, and $\tau := \sigma^{-1} \in \Sigma_{2n}$ is the inverse permutation. Denote the set of symbols $(a_1, a_1^{-1}, \dots, a_n, a_n^{-1})$ by $(w_{\tau(1)}, \dots, w_{\tau(2n)})$. Thus $w_{\tau(j)} = a_{k(j)}^{\varepsilon(j)}$ where $k(j) := \lceil j/2 \rceil$, $\varepsilon(j) := (-1)^{j-1}$, $1 \leqslant j \leqslant 2n$. A *quadratic word* is the formal product

$$W := w_1 \dots w_{2n} = \prod_{j=1}^{2n} w_j = \prod_{j=1}^{2n} a_{k(\sigma(j))}^{\varepsilon(\sigma(j))}.$$

Consider the closed unit 2-disk $D \subset \mathbb{C}$ centred at the origin, with $2n$ marked points $v_\ell = e^{\pi i \ell/n} \in \partial D$, $1 \leqslant \ell \leqslant 2n$. Consider the cell decomposition of D into $2n$ vertices $v_1, \dots, v_{2n}$, $2n$ directed open edges and the open 2-cell $\operatorname{int} D := D \setminus \partial D$. The edges are divided into pairs $\alpha_k(t) := e^{\pi i(\tau(2k-1)+t)/n}$ and $\widehat{\alpha}_k(t) := e^{\pi i(\tau(2k)+1-t)/n}$ $(0 \leqslant t \leqslant 1)$ corresponding to the letters a_k and a_k^{-1} of W, $1 \leqslant k \leqslant n$. Denote by β_k the directed curve on D formed by the radii that terminate at the centres of the edges α_k and $\widehat{\alpha}_k$, directed from $\alpha_k(1/2)$ to $\widehat{\alpha}_k(1/2)$, $1 \leqslant k \leqslant n$. Consider the topological space

$$C = C_W := D/\{\alpha_k(t) \sim \widehat{\alpha}_k(t),\ 0 \leqslant t \leqslant 1\}_{k=1}^n$$

with quotient topology. It is a connected orientable closed surface (of $\dim_{\mathbb{R}} C = 2$), with the induced cell decomposition having $s := n+1-2g$ vertices, n directed edges identified with $\alpha_1, \dots, \alpha_n$, and one 2-cell. Here g denotes the genus of C. Denote by $C^r \subset C$ the r-skeleton of this cell decomposition, $0 \leqslant r \leqslant 2$. Thus

$$S = S_W := C \setminus C^0$$

is a connected orientable genus-g surface with s punctures. If the graph $\cup_{k=2g+1}^n \alpha_k$ is a spanning tree of the graph C^1 (as can be achieved by renumbering the symbols $a_1, \dots, a_n$) then the quadratic word W will be called *combinatorial data* of genus (g, s).

2.2. Geometrical data. Suppose $U_1 \subset \mathbb{C}$ is an open subset, and

$$\mathbf{J} = (J_1, \dots, J_n) : U_1 \to (\mathbb{C} \setminus \{0\})^n, \qquad J_0 : U_1 \to \mathbb{C}$$

are continuous maps. The triple $(U_1, \mathbf{J}, J_0)$ is called *geometrical data* with respect to the data W.

2.3. Combinatorial-geometrical data. Let us describe a natural geometrical object associated to the CG-data $(W; U_1, \mathbf{J}, J_0)$, namely a 2-parameter family of closed planar polygonal lines with certain pairing of sides.

The group $H_1(S) \cong \mathbb{Z}^n$ admits the basis $\{[\beta_k]\}_{k=1}^n$. The group $H_1(C, C^0) \cong \mathbb{Z}^n$ admits the bases $\{[\alpha_k]\}_{k=1}^n$ and $\{[\widetilde{\alpha}_k]\}_{k=1}^n$, where $\widetilde{\alpha}_k := \beta_k$ for $1 \leqslant k \leqslant 2g$, $\widetilde{\alpha}_k := \alpha_k$ for $2g+1 \leqslant k \leqslant n$. Here $[\alpha_k], [\beta_k]$ denote the homology classes of the curves

α_k, β_k in appropriate homology groups. Consider the coordinate isomorphisms $\boldsymbol{\mu} = (\mu_1, \dots, \mu_n) : H^1(S; \mathbb{C}) \to \mathbb{C}^n$ and

$$\mathbf{m} = (m_1, \dots, m_n),\ \ \widetilde{\mathbf{m}} = (\widetilde{m}_1, \dots, \widetilde{m}_n) : H^1(C, C^0; \mathbb{C}) \to \mathbb{C}^n$$

with respect to these bases. Consider natural homomorphisms

$$H_1(S) \xrightarrow{\ i\ } H_1(C) \xrightarrow{\ p\ } H_1(C, C^0).$$

Define the linear map $\mathbf{A} = (A_1, \dots, A_{2n}) : \mathbb{C}^n \to \mathbb{C}^{2n}$ by the rule

$$A_\ell(\mathbf{z}) := \sum_{j=1}^{\ell-1} \varepsilon(\sigma(j)) z_{k(\sigma(j))}, \quad \mathbf{z} = (z_1, \dots, z_n) \in \mathbb{C}^n,\ 1 \leqslant \ell \leqslant 2n.$$

The homomorphism $i^* p^* : H^1(C, C^0; \mathbb{C}) \to H^1(S; \mathbb{C})$ has coordinate presentation $\mathbf{T} = (T_1, \dots, T_n) = \boldsymbol{\mu} i^* p^* \mathbf{m}^{-1} : \mathbb{C}^n \to \mathbb{C}^n$ of the form

$$T_k(\mathbf{m}) = A_{\tau(2k)+1}(\mathbf{m}) - A_{\tau(2k-1)}(\mathbf{m}) = \sum_{\ell=1}^{n} \varphi_{\ell k} m_\ell, \quad 1 \leqslant k \leqslant n.$$

Here $\varphi_{\ell k} := \langle [\beta_\ell], [\beta_k] \rangle$ is the intersection index of the cycles $[\beta_\ell]$ and $[\beta_k]$ in $H_1(S)$. Thus $\operatorname{rank}(i^* p^*) = \operatorname{rank}(p^*) = 2g$, and $\widetilde{m}_k = T_k(\mathbf{m})$ for $1 \leqslant k \leqslant 2g$, $\widetilde{m}_k = m_k$ for $2g + 1 \leqslant k \leqslant n$.

Consider the 2-parameter family of the closed $2n$-gonal lines on $\mathbb{C}$ with consecutive vertices

$$A_1(\mathbf{J}(\xi)) + J_0(\xi), \dots, A_{2n}(\mathbf{J}(\xi)) + J_0(\xi) \in \mathbb{C}, \qquad \xi \in U_1,$$

with parameter ξ, more precisely the 2-parameter family of the planar closed paths $\theta_{W;\mathbf{J}(\xi),J_0(\xi)} : \partial D \to \mathbb{C}$ formed by the segments

$$\theta_{W;\mathbf{J}(\xi),J_0(\xi)}(e^{\pi i(\ell+t)/n}) = (1-t) A_\ell(\mathbf{J}(\xi)) + t A_{\ell+1}(\mathbf{J}(\xi)) + J_0(\xi), \quad 0 \leqslant t \leqslant 1,$$

$1 \leqslant \ell \leqslant 2n$, $\xi \in U_1$, where $A_{2n+1} := A_1$.

2.4. Topological data. Take a point $\xi_0 \in U_1$. A continuous map $\Theta_{\xi_0} : D \to \mathbb{C}$ is called *an extension* of the closed $2n$-gonal line $\theta = \theta_{W;\mathbf{J}(\xi_0),J_0(\xi_0)}$ if $\Theta_{\xi_0}|_{D \setminus \{v_\ell\}_{\ell=1}^{2n}}$ is an orientation-preserving immersion and $\Theta_{\xi_0}|_{\partial D} = \theta$. So, an extension may have branching at the vertices v_ℓ. In general, an extension is not unique up to homeomorphisms of D identical on ∂D (e.g. the octagon with consecutive vertices $-2-i$, i, $2-i$, $-1/2$, $2+i$, $-i$, $-2+i$, $1/2$ has inequivalent extensions [7]). Any extension Θ_{ξ_0} of θ can be included into a family of extensions $\Theta_\xi : D \to \mathbb{C}$ of the closed $2n$-gonal lines $\theta_{W;\mathbf{J}(\xi),J_0(\xi)}$, $\xi \in U$, such that the map

$$\Theta = \Theta_{W;U,\mathbf{J},J_0;\theta} : U \times D \to \mathbb{C}, \qquad (\xi, d) \mapsto \Theta_\xi(d),$$

is continuous, where $U \subset U_1$ is a small enough simply-connected neighbourhood of ξ_0. Consider the set $[\Theta]$ of maps obtained by composing Θ with homeomorphisms of $U \times D$ that are identical on $U \times \partial D$ and preserve each fibre of the projection $U \times D \to U$. The set $[\Theta]$ is called *topological data* with respect to the data $(W; U, \mathbf{J}, J_0)$.

§ 3. "Exact" geometrical data and canonical Liouville fibrations

Consider the topological 4-manifold

$$M_{W;U} := U \times S_W = U \times S$$

and its open subsets

$$M_0 := U \times (C \setminus C^1) \approx U \times \operatorname{int} D, \qquad M_k := U \times (p_W(U_k^+ \cup U_k^-)), \quad 1 \leqslant k \leqslant n.$$

Here $p_W : D \setminus \{v_\ell\}_{\ell=1}^{2n} \to S_W$ is the projection, U_k^+ and U_k^- are small disjoint neighbourhoods of the open edges α_k and $\widehat{\alpha}_k$ (respectively) in $D \setminus \{v_\ell\}_{\ell=1}^{2n}$. Consider the following immersions of these subsets into $U \times \mathbb{C} \subset \mathbb{C}^2$:

$$(\xi, \Theta^0) : M_0 \looparrowright U \times \mathbb{C}, \qquad (\xi, \Theta^k) : M_k \hookrightarrow U \times \mathbb{C}, \quad 1 \leqslant k \leqslant n,$$

where $\xi : M_{W;U} \to U$ is the projection, $\Theta^0 := \Theta|_{M_0}$; the map $\Theta^k : M_k \to \mathbb{C}$, $1 \leqslant k \leqslant n$, is defined by the rules

$$\Theta^k(\xi, p_W(d)) := \begin{cases} \Theta(\xi, d) & \text{if} \quad d \in U_k^+, \\ \Theta(\xi, d) - (0, T_k(\mathbf{J}(\xi))) & \text{if} \quad d \in U_k^-, \end{cases}$$

$(\xi, d) \in M_k$. Denote by $\mathcal{A}$ the atlas on $M_{W;U}$ formed by all coordinate charts with local coordinates (ξ, Θ^k) on M_k, $0 \leqslant k \leqslant n$.

THEOREM 1. *The following ("exactness") conditions are equivalent:*

(i) *the atlas $\mathcal{A}$ on $M_{W;U}$ is smooth, and the symplectic 2-form $\{\operatorname{Re}(d\xi \wedge d\Theta^k)\}_{k=0}^n$ on $(M_{W;U}, \mathcal{A})$ is well-defined;*

(ii) *$((M_{W;U}, \mathcal{A}), \{\operatorname{Re}(d\xi \wedge d\Theta^k)\}_{k=0}^n, \xi)$ is a Liouville fibration;*

(iii) *the 1-forms $\operatorname{Re}(T_k(\mathbf{J}(\xi))d\xi)$ with $1 \leqslant k \leqslant 2g$ (equivalently, with $1 \leqslant k \leqslant n$) on U are smooth and closed;*

(iv) *the surfaces $\{(\xi, T_k(\mathbf{J}(\xi))) \mid \xi \in U\}$ with $1 \leqslant k \leqslant 2g$ (equivalently, with $1 \leqslant k \leqslant n$) are smooth and Lagrangian in $(U \times \mathbb{C}, \operatorname{Re}(d\xi \wedge d\Theta))$.*

Suppose

$$\operatorname{Re}(T_k(\mathbf{J}(\xi))d\xi) = 2\pi\, dI_k(\xi), \qquad \xi \in U, \quad 1 \leqslant k \leqslant 2g,$$

for some smooth map $\mathbf{I} = (I_1, \dots, I_{2g}) : U \to \mathbb{R}^{2g}$. Thus the condition (iii) of theorem 1 holds. Put

$$\mathbf{S} := (\mathbf{K}, J_0) \in C^0(U, \mathbb{C}^s) \quad \text{where} \quad \mathbf{K} := (J_{2g+1}, \dots, J_n) \in C^0(U, \mathbb{C}^{s-1}),$$

thus $(T_1 \circ \mathbf{J}, \dots, T_{2g} \circ \mathbf{J}, \mathbf{S}) = (\widetilde{\mathbf{m}} \circ \mathbf{m}^{-1} \circ \mathbf{J}, J_0)$. Let us associate the triple $(U, d\mathbf{I}, \mathbf{S})$ to such geometrical data $(U, \mathbf{J}, J_0)$. By misuse of language, the triple $(U, d\mathbf{I}, \mathbf{S})$ will be called *exact geometrical data with respect to W*.

By theorem 1, any *exact CGT-data* $(W; U, d\mathbf{I}, \mathbf{S}; [\Theta])$ determine a unique (up to equivalence) Liouville fibration, which is denoted by $L_{W;U,d\mathbf{I},\mathbf{S};[\Theta]}$ and called a *canonical Liouville fibration.*

THEOREM 2. *For any pair of canonical Liouville fibrations* $L_{W;U,d\mathbf{I},\mathbf{K},J_0;[\Theta]}$ *and* $L_{W;U,d\mathbf{I}^*,\mathbf{K}^*,J_0^*;[\Theta^*]}$ *with the same data* $(W;U)$, *the following conditions are equivalent:*

(i) *for some* $\xi_0 \in U$, *there exists an isomorphism* $h : M_{W;U} \to M_{W;U}$ *of these fibrations identical on the set* $\{\xi_0\} \times (C_W^1 \setminus C_W^0)$;

(ii) $d\mathbf{I} = d\mathbf{I}^*$ *and* $\mathbf{K} = \mathbf{K}^*$; $\Theta_\xi = \Theta_\xi^* \circ h_\xi + J_0(\xi) - J_0^*(\xi)$ *for some family of homeomorphisms* $h_\xi : D \to D$ *identical on* ∂D, $\xi \in U$; *the 1-form* $\mathrm{Re}((J_0(\xi) - J_0^*(\xi))d\xi)$ *on* U *is smooth and closed.*

§ 4. The Liouville-type theorem

For any Liouville fibration (M^4, ω, F), denote by V_1, V_2 the Hamiltonian vector fields with the Hamiltonian functions f_1, f_2. On each regular fibre $T_\xi := F^{-1}(\xi)$, consider the flat Riemannian metric g_ξ inverse to the bi-vector field $(V_1 \otimes V_1 + V_2 \otimes V_2)|_{T_\xi}$. Denote by $\overline{T_\xi}$ the completion of T_ξ with respect to this Riemannian metric.

THEOREM 3. *Suppose that a Liouville fibration* (M^4, ω, F) *is topologically locally trivial; moreover any its fibre* T_ξ *is regular and connected, the completion* $\overline{T_\xi}$ *of* T_ξ *is compact and* $0 < |\overline{T_\xi} \setminus T_\xi| < \infty$, $\xi \in F(M^4) \subset \mathbb{R}^2 \cong \mathbb{C}$.

Then any point $\xi \in F(M^4)$ *has a neighbourhood* U *such that the Liouville fibration* $(F^{-1}(U), \omega, F)$ *is isomorphic to a canonical Liouville fibration* $L_{W;U,d\mathbf{I},\mathbf{S};[\Theta]}$, *for some exact CGT-data of genus* (g, s), *where* $s := |\overline{T_\xi} \setminus T_\xi|$, $g := 1 - (\chi(T_\xi) + s)/2$. *The flat Riemannian metric* g_ξ *on any fibre* T_ξ *has a conical singularity at any puncture, where all cone angles are integer multiples of* 2π.

§ 5. Examples via complex polynomials and their Newton polygons

Consider a non-constant polynomial $f(z, w) = \sum_{\ell,m \geqslant 0} a_{\ell m} z^\ell w^m$ in two complex variables. H. Flaschka [1] and A. I. Shafarevich observed that the triple $(\mathbb{C}^2, \mathrm{Re}(dz \wedge dw), f)$ is a Liouville fibration. The set Σ_f of critical values of f is known to be finite [8; §2]. The *Newton polygon* of f is the convex hull of its "support":

$$\Delta_f := \operatorname{conv}\left\{(\ell, m) \in \mathbb{Z}^2 \mid a_{\ell m} \neq 0\right\} \subset \mathbb{R}^2.$$

For any side e of Δ_f, denote $f_e(z, w) := \sum_{(\ell,m) \in e} a_{\ell m} z^\ell w^m$. The polynomial f is called *weakly nondegenerate* with respect to Δ_f if, for any side e of Δ_f that does not lie on coordinate axes and for any point $(z, w) \in (\mathbb{C} \setminus \{0\})^2$ with $f_e(z, w) = 0$, one has $df_e(z, w) \neq 0$.

THEOREM 4. *Suppose a polynomial* $f = f(z, w)$ *is weakly nondegenerate with respect to its Newton polygon* Δ_f, *moreover* $\Delta_{1+z^\ell+w^m} \subseteq \Delta_f \subseteq \Delta_{(1+z^\ell)(1+w^m)}$ *for some* $\ell, m \in \mathbb{N}$. *Let* $g > 0$ *be the number of integer points in the interiour of* Δ_f, *and* $s - 1$ *be the number of integer points of* $\partial\Delta_f$ *that do not lie on coordinate axes. Then the Liouville fibration* $(\mathbb{C}^2 \setminus f^{-1}(\Sigma_f), \mathrm{Re}(dz \wedge dw), f)$ *satisfies all the hypothesis of theorem* 3.

Moreover, the exact CGT-data $(W; U, d\mathbf{I}, \mathbf{S}; [\Theta])$ *of any corresponding canonical Liouville fibration have genus* (g, s) *and satisfy the following:*

(i) *there exist functions* $I_{2g+1}, \ldots, I_n, I_0 \in C^\infty(U, \mathbb{R})$ *such that* $\mathrm{Re}(J_0(\xi)d\xi) = dI_0(\xi)$, $\mathrm{Re}(J_k(\xi)d\xi) = dI_k(\xi)$, $2g+1 \leqslant k \leqslant n$, $\xi \in U$, *where* $\mathbf{S} = (J_{2g+1}, \ldots, J_n, J_0) \in$

$C^0(U,\mathbb{C}^s)$; moreover the functions $I_0,\dots,I_n\in C^\infty(U,\mathbb{R})$ are real parts of some holomorphic functions on $U\subset\mathbb{C}$;

(ii) *for any $\xi\in\mathbb{C}\setminus\Sigma_f$, there exists a bijection between the punctures of T_ξ and the couples of neighbour integer points $A,B\in\partial\Delta_f$ that do not belong to the same coordinate axis, satisfying the following condition: the cone angle of the flat Riemannian metric g_ξ at the puncture corresponding to $\{A,B\}$ equals $4\pi s_{\{A,B\}}$ where $s_{\{A,B\}}$ is the area of the triangle $A,B,(1,1)$.*

Theorem 4 can be proved by using [8, 9, 10, 11]. Its analogues for hyperelliptic polynomials are given in [12, 10, 11].

References

[1] *Flaschka H.* A remark on integrable Hamiltonian systems // Physics Letters A. 1988. V. 131. № 9. P. 505–508.

[2] *Bolsinov A. V., Fomenko A. T.* Some actual unsolved problems on topology of integrable Hamiltonian systems // In book “Topological methods in theory of Hamiltonian systems”. M.: Izd-vo Faktorial, 1998. P. 5–23 (in Russian).

[3] *Bates L., Cushman R.* Complete integrability beyond Liouville-Arnol'd // Rep. Math. Phys. 2005. V. 56. № 1. P. 77–91.

[4] *Kontsevich M. L., Zorich A. V.* Connected components of the moduli spaces of Abelian differentials with prescribed singularities // Inventiones mathematicae. 2003. V. 153. № 3. P. 631–678.

[5] *Novikov S. P.* Topology of generic Hamiltonian foliations on Riemann surfaces // Mosc. Math. J. 2005. T. 5. № 3. C. 633—667.

[6] *Bufetov A. I.* Decay of correlations for the Rauzy-Veech-Zorich induction map on the space of interval exchange transformations and the central limit theorem for the Teichmüller flow on the moduli space of abelian differentials // J. Amer. Math. Soc. 2006. V. 19. № 3. P. 579–623.

[7] *Poénaru V.* Extension des immersions en codimension one (d'aprés S. Blank) // Séminaire Bourbaki. 1968. № 342. P. 473–505.

[8] *Kudryavtseva E. A., Lepskii T. A.* Integrable Hamiltonian systems with incomplete flows and Newton's polygons // Contemp. Probl. Mathem. Mechan. 2011. V. VI. № 3. P. 42–55 (in Russian).

[9] *Khovanskii A. G.* Newton polyhedra, and the genus of complete intersections // // Funct. Anal. Appl. 1978. V. 12. № 1. P. 38–46.

[10] *Kudryavtseva E. A., Lepskii T. A.* The topology of Lagrangian foliations of integrable systems with hyperelliptic Hamiltonian // Sb. Math. 2011. V. 202. № 3–4. P. 373–411.

[11] *Kudryavtseva E. A., Lepskii T. A.* Topology of foliation and the Liouville theorem for integrable systems with incomplete flows // Sb. Tr. Sem. Vekt. Tenz. Analizu. 2012. V. XVII. P. 104–148 (in Russian).

[12] *Lepskii T. A.* Incomplete integrable Hamiltonian systems with complex polynomial Hamiltonian of small degree // Sb. Math. 2010. V. 201. № 10. P. 1511–1538.

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